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INTEGRAL ESTIMATES IN NON-LINEAR OSCILLATORY SYSTEMS†

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A procedure for constructing integral estimates for the wave equation with a non-linearity concentrated in a certain local region of the system in question is presented. To obtain these estimates a knowledge of the solution of the linear homogeneous differential equation and the dynamics of wave propagation in the system is required. The accuracy of the estimates depend considerably on the nature of the external loads, their intensity and duration. It makes sense to consider this type of estimate for processes whose duration is comparable with the time taken for the wave to traverse the system, which is characteristic, for example, for explosive and shock phenomena, and also dynamic fracture. An example of the construction of wave estimates for a simple mechanical system is considered.

1. FORMULATION OF THE PROBLEM

Many problems in the mechanics and physics of oscillations lead to an equation of the form [1]

$$\rho\ddot{u} = L[u]+q, \quad L[u] = \operatorname{div}(p \operatorname{grad} u) \tag{1.1}$$

with initial conditions

$$u\big|_{t=0} = U_0(x), \quad \dot{u}\big|_{t=0} = \dot{U}_0(x), \quad x \in V \tag{1.2}$$

and mixed boundary conditions

$$\alpha u + \beta\, \partial u / \partial n\big|_{\partial V} = 0, \quad t>0 \tag{1.3}$$

where the unknown function $u(x, t)$ depends on n spatial coordinates $x = (x_1, x_2, \ldots, x_n)$ (usually $n = 1, 2$, or 3) and the time t, and the coefficients ρ and p are governed by the properties of the medium and are independent of time. In view of their physical meaning we will assume that $\rho(x) > 0$ and $p(x) > 0$; $V \subset R^n$ is the region where the process occurs, and ∂V is its boundary, which we will assume to be a piecewise-smooth surface, α and β are independent of t, and ∂V_0 is the part of the surface ∂V where $\alpha(x) > 0$ and $\beta(x) > 0$ simultaneously.

The free term q expresses the intensity of the external force on the system. If it depends solely on the spatial coordinates and the time, the solution of the problem can be found using well-known approaches [1].

The search for a solution becomes considerably more complicated if the intensity of the external forces q depends non-linearly on the solution itself and on the partial derivatives of this solution with respect to time and the spatial coordinates. Approximate solutions and estimates therefore become important here.

Below we present a procedure for constructing such estimates when the function $q = q(x, t, u, \ldots)$ is concentrated in a certain region $D \subset V$ (Fig. 1), while outside this region $q = 0$.

The estimate is constructed using the solution $u_0(x, t)$ of the linear hyperbolic homogeneous differential equation corresponding to (1) when $q = 0$, with the same initial and boundary conditions (1.2) and (1.3). This solution we will henceforth assume to be known. We will also assume that, for all the coefficients and functions considered in this paper, all the necessary smoothness conditions are satisfied.

2. INTEGRAL ESTIMATES

We will assume that the non-linear problem considered is well posed and $u(x, t)$ is its solution. We introduce the energy integral of the system [1]

$$I(\tau) = \frac{1}{2}\int_V i_x(\tau)dx + \frac{1}{2}\int_{\partial V_0} p\frac{\alpha}{\beta}u^2 ds \tag{2.1}$$

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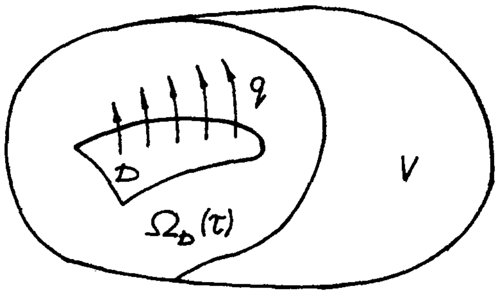

Fig. 1.

$$i_x(\tau) = \rho \dot{u}^2 + p|\operatorname{grad} u|^2 \tag{2.2}$$

which is the sum of the kinetic and potential energy of the system at the instant of time $t = \tau$, $I(\tau) \geqslant 0$ The following relation then holds [1]

$$A(\tau) = I(\tau) - I(0) \quad \left(A(\tau) = \int_0^\tau \int_D q\dot{u}\, dx\, dt \right) \tag{2.3}$$

which has an explicit physical meaning.

For the instant of time τ we will introduce the wave volume of the external actions $\Omega_D(\tau)$ as the set of all points x ($x \in V$) encompassing perturbations due to the action of the external forces (which, by convention, are localized in the region D) in the time interval from $t = 0$ to $t = \tau$. This can be done since the velocity of propagation of the perturbations in the system (outside the region D) is $C = (p/\rho)^{1/2}$ and is finite.

We will represent $I(\tau)$ in the form of the sum of two terms

$$I(\tau) = I_I(\tau) + I_N(\tau) \tag{2.4}$$

where $I_I(\tau)$ is the total energy of the wave volume $\Omega_D(\tau)$ and $I_N(\tau)$ is the total energy of the remaining volume, unperturbed by the external forces, of the system considered. Since $I_I(\tau) \geqslant 0$ and assuming that $I_I(\tau) = 0$, we obtain the following limit

$$A(\tau) \geqslant I_N(\tau) - I(0) \tag{2.5}$$

A feature of the last inequality is the fact that its right-hand side can be written in terms of the solution $u_0(x, t)$ of the homogeneous differential equation, since this solution is identical with the solution of the non-linear problem outside the wave volume of the external forces (more correctly, outside the region of influence D) [1], where the points of the system "know" nothing of the non-linear external action in the time interval from $t = 0$ to $t = \tau$.

Hence, inequality (2.5) is the lower limit of the work done by the external forces for the non-linear system in the time interval from zero to τ, or, in other words, the maximum value of the work which the system can perform in this time interval.

3. THE WAVE VOLUME OF EXTERNAL ACTIONS

For the integral estimate (2.5) we need to know the wave volume of the external forces, in addition to the solution of the homogeneous differential equation.

In a homogeneous medium (ρ and p are independent of the spatial coordinates x) the problem of finding the wave volume (more correctly, both for the regions of influence and dependence also reduces to a geometrical problem, namely, the construction of the external envelope of the spheres $S(\xi, C\tau)$, when ξ runs through ∂D (Huygens' principle). This external envelope will be the boundary of the wave volume $\partial\Omega_D(\tau)$. Another method is to construct the wave rays.

The dependence of the characteristics of the medium on the spatial coordinates changes the shape of the wave beams, but they remain perpendicular to the wave front. Here we can also use Huygens' principle to obtain the wave volume or directly integrate the system of ordinary non-linear differential equations for determining the trajectory of a wave ray [2]. Here it is necessary to take some care since the dependence of the medium characteristics on the spatial coordinates may lead to different wave effects, which must be taken into account when constructing the wave volume (the formation of shadow zones, waveguides, caustics, etc.).

4. REFINED INTEGRAL ESTIMATES

More accurate upper estimates of the value of the work of the system $W(\tau) = -A(\tau)$ can be obtained by a non-trivial estimate of the total energy of the wave volume $I_1(\tau)$ (we recall that initially this was assumed to be zero). We will consider one of the possible schemes for constructing such estimates.

For an arbitrary point ξ of the wave volume of the external forces $\Omega_D(\tau)$, lying outside the region D, we will consider the total energy density $i_\xi(\tau)$ at the instant of time $t = \tau$, defined by (2.2). We will represent the function $u = u(\xi, \tau)$ as the solution of homogeneous differential equation (1.1) with the following initial conditions

$$u\Big|_{t=\eta} = U_\eta(x), \quad \dot{u}\Big|_{t=\eta} = \dot{U}_\eta(x); \quad x \in \Omega_\xi(\chi), \quad \chi = \tau - \eta \tag{4.1}$$

Here $\Omega_\xi(\chi)$ is a region of the dependence ξ which should have no intersections with the region D, which can always be achieved by an appropriate choice of the instant of time η $(0 < \eta < \tau)$. If necessary (if $\Omega_\xi(\chi)$ reaches the boundary of the system), the boundary conditions on the boundary ∂V (1.3) close the formulation of the problem. In the case of a uniform medium with $n = 1, 2, 3$, these representations are given d'Alembert's, Poisson's and Kirchhoff's relations, respectively.

The boundary of the wave volume of the external forces $\partial\Omega_D(\eta)$ at the instant of time $t = \eta$ divides $\Omega_\xi(\chi)$ into two regions (Fig. 2): in the first $U_\eta(x) = u_0(x, \eta)$ and $\dot{U}_\eta(x) = \dot{u}_0(x, \eta)$ are identical with the unknown solution of the homogeneous differential equation considered with initial and boundary conditions (1.2) and (1.3); in the second the initial conditions $U_\eta(x) = u_n(x, \eta)$ and $\dot{U}_\eta(x) = \dot{u}_n(x, \eta)$ are perturbed by the external non-linear forces.

We will seek the lowest value of the total energy density functional using the corresponding representation of the solution $u = u(\xi, \tau)$

$$\mathrm{i}_\xi(\tau, \eta) = \inf_{u_n, \dot{u}_n} i_\xi(\tau) \tag{4.2}$$

provided the initial conditions are specified in a certain region by the known solution $u_0(x, t)$, and are arbitrary outside this region. An additional condition is the continuity of the function $u(x, t)$ on the boundary of the wave volume $\partial\Omega_D(\eta)$

$$u_n(\partial\Omega_D(\eta), \eta) = u_0(\partial\Omega_D(\eta), \eta) \tag{4.3}$$

which follows from physical considerations.

The fact that this extremal problem is ill-posed may be due to two factors [1]: either the functional $i\xi(\tau)$ has no lower limit on certain sets M_u and $M_{\dot{u}}$, $u_n \in M_u$, $\dot{u}_n \in M_{\dot{u}}$ (usually M_u and $M_{\dot{u}}$ are sets of differentiable functions), or it has a lower limit but its lower boundary is not reached on M. The functional $i\xi(\tau)$ obviously has a lower limit (for example, the zero value $i\xi(\tau) \geqslant 0$) and since a knowledge of the minimum element for this extremal problem is not required, it is well-posed and its solution can be found by well-known approaches [3].

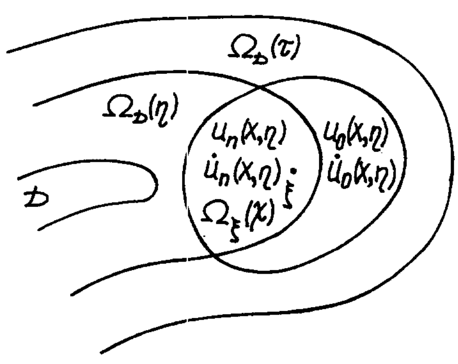

Fig. 2.

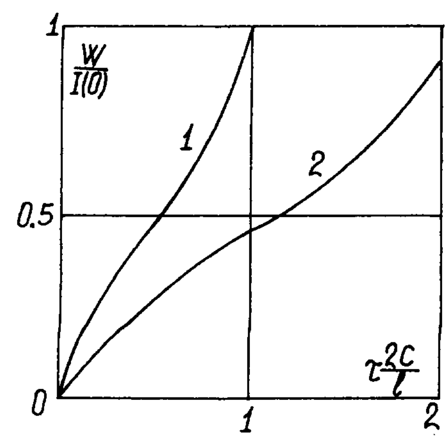

Fig. 3.

This solution also depends on the parameter η, and hence, from the parametric set of lower limits obtained, it is logical to choose the largest one, since it will be closest to the actual one, i.e.

$$\mathrm{i}_{\xi}(\tau)=\max_{\eta}\ \mathrm{i}_{\xi}(\tau,\eta) \tag{4.4}$$

If one succeeds in obtaining a non-trivial solution for $i\xi(\tau)$, then, by integrating over the whole wave volume (apart from the region D), we obtain a non-trivial estimate of the total energy of the wave volume

$$I_l(\tau)\geqslant I_l(\tau)=\int_{\Omega_D(\tau)-D}\mathrm{i}_{\xi}(\tau)d\xi \tag{4.5}$$

and the wave limit can be written as follows:

$$A(\tau)\geqslant I_l(\tau)+I_N(\tau)-I(0) \tag{4.6}$$

5. EXAMPLE

Consider the inhomogeneous one-dimensional wave equation

$$\rho\ddot{u}-p\partial^2u/\partial x^2=q \tag{5.1}$$

which describes, for example, the oscillations of a uniform string (ρ and p are constants) with clamped ends

$$u(0)=u(l)=0 \tag{5.2}$$

where l is the length of the string.

To fix our ideas, we will specify the following initial conditions

$$u\Big|_{t=0}=U_0(x)=a\sin x \text{ and } \dot{u}\Big|_{t=0}=\dot{U}_0(x)=0 \tag{5.3}$$

The external forces q acting on the system when $t>0$, are concentrated at one point $x=l/2$ of the middle of the string and depend non-linearly on the system parameters.

The solution of the linear homogeneous differential equation corresponding to (5.1) with the same boundary and initial conditions (5.2) and (5.3) describes a standing wave with an oscillation frequency $\omega=(\pi/l)(p/\rho)^{1/2}$.

For this example we obtain an upper limit of the work which the system can perform as a function of time.

By (2.5) the limit of the work which the system $W(\tau)=-A(\tau)$ can carry out will have the form

$$W(\tau)\leqslant p\frac{a^2\pi}{4l}\left(2\omega\tau+\frac{1}{2}\sin 2\omega\tau\cos 2\omega\tau\right) \tag{5.4}$$

One can improve the upper values (5.4) of the work which the system can perform at the cost of a non-trivial estimate of the total energy of the wave volume.

For the point $\xi\neq 1/2l$ and the instant of time τ we introduce the total energy density by (2.2).

Solving the variational problem (4.1)–(4.4) for the example considered, we obtain the lower limit of the total energy of the wave volume, which is then substituted into the estimate of the work done by the external forces (4.6).

The results of the calculations are shown in Fig. 3, where curve 1 corresponds to the wave estimates (5.4) and curve 2 corresponds to the refined wave estimates of the upper values of the work of the system as a function of time.

A more interesting example of the use of wave estimates to solve problems of brittle dynamic fracture was considered in [4].

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